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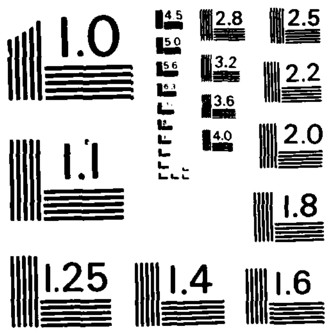
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QUALITY ASSESSMENT AND CONTROL OF FINITE ELEMENT SOLUTIONS

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ABSTRACT

Status and some recent developments in the techniques for assessing the reliability of finite element solutions are summarized. Discussion focuses on a number of aspects including: the major types of errors in the finite element solutions; techniques used for **a posteriori** error estimation and the reliability of these estimators; the feedback and adaptive strategies for improving the finite element solutions; and postprocessing approaches used for improving the accuracy of stresses and other important engineering data. Also, future directions for research needed to make error estimation and adaptive improvement practical are identified.

1. INTRODUCTION

The finite element method has become the main tool in computational mechanics and to date the method is used for solving a large class of engineering problems which are stated in terms of differential, pseudo-differential, integral or integro-differential equations. The success of the finite element method is manifested by the development of over five hundred user-oriented finite element program systems, and over two hundred pre- and postprocessing packages (Ref.1). There are over 20,000 finite element users worldwide who are estimated to spend about $500 million annually on finite element analyses. The literature on the subject is nearly overwhelming and to date there are over two hundred monographs and conference proceedings published on various aspects of the finite element technology (Ref. 2). A review of some recent developments in finite element method is contained in survey papers (see, for example, Refs. 3 and 4) and a state-of-the-art monograph (Ref. 5). Despite the significant advances made on the theory and algorithmic tools of the finite element method, the selection of the finite element model for a particular problem is largely based on intuition and experience gained from solving similar problems. Moreover, the assessment of the reliability of the finite element solution continues to be the most difficult aspect of the finite element analysis. Some aspects of the errors in finite element computations are discussed in Refs. 6 and 7. Error estimation and control is now at the "cutting edge" of the finite element technology.

In recent years considerable interest has been shown in the development of reliable error estimates as well as feedback procedures (or adaptive strategies) by which a required accuracy of the finite element solution can be most economically reached. The number of publications on the subject has been steadily increasing and, although two symposia proceedings have been published on the subject (Refs. 8 and 9), there is a need to broaden awareness among practicing engineers and research workers about the status and recent developments in various aspects of quality assessment and control of the finite element solutions. This paper is an attempt to fill this void. Specifically, the objectives of this paper are: a) to review and assess the current techniques used for error estimation and adaptive improvement of the finite element solutions; and b) to identify future directions of research needed to make error estimation and adaptive improvement practical.

The contents of the paper are arranged as follows: in section 2, the major types of errors in a finite element solution are listed; in section 3, the available technology for estimating the discretization errors is reviewed. Then the different strategies for adaptive improvement of the finite element solution are discussed in section 4. In section 5 the postprocessing techniques used for improving the quality of stresses and other important engineering data are reviewed. The future directions of research on quality assessment and control are identified in section 6.

There is a large literature available. The cited references are selected for illustrating the ideas presented and are not necessarily the only significant contributions to the subject. The discussion in this paper is kept on a descriptive level and for all the mathematical details the reader is referred to the cited literature.

2. MAJOR ERRORS IN THE FINITE ELEMENT SOLUTION

In general, the reliability of the finite element solutions to engineering problems is influenced by the following factors (see Refs. 6 and 7):

a) **Reliability of the mathematical model** which describes the real structure in mathematical terms. The stochastic formulation is one of the possible mathematical formulations of the engineering problem. However, deterministic formulations are more commonly used.

b) **Errors and uncertainties in the input information** (of the mathematical model). These include uncertainties in material, geometry; boundary conditions; loading environment; and the characteristics of the probability fields in the stochastic formulation.

c) **Discretization errors** which are caused by the numerical discretization of the continuous mathematical model, as well as by the truncation of infinite processes (e.g., iterative procedures and summation of infinite series).

d) **Roundoff errors** which occur in the implementation of numerical algorithms on computers with finite precision (limitation in representing real numbers due to the finiteness of the computer word length).

An assessment of the reliability of the mathematical model requires the identification of the range of validity of the mathematical theory used in describing the model, and the effect of violating some of the basic assumptions of the theory on the response predictions. Examples of these issues are provided by the range of validity of dimensional reduction when applied to plates with rapidly varying thickness (i.e., the reduction from three-dimensional continuum theory to two-dimensional plate theory - Refs. 10 and 11); the applicability of the linear elasticity theory in the neighborhood of a crack tip; and the effect of "pseudo-supports" on the response of structural members (Ref. 7).

The stochastic formulation, which is widely used in dynamics, leads to a deterministic formulation for the mean and correlation functions of the solution (see Refs. 12 to 20). Monte Carlo methods are sometimes valuable tools for the mathematical treatment of the uncertainties in the problem parameters. As an alternative to stochastic modeling, an assessment of the effect of the uncertainties in the input information on the response can be made by evaluating the sensitivity of the response of the system to each of the input parameters.

The discretization error represents the difference between the exact and the approximate (finite element) solutions for the mathematical model. The determination of the actual discretization error requires the availability of the exact solution, which is rarely known in practical problems. However, even when the exact solution is not available, it is possible to construct **quantitative estimates** for the discretization error and to determine the rate of change of the error as the number of degrees of freedom in the finite element model increases.

Since discretization errors are functions, error norms are usually used to measure the size of these errors. For a brief discussion of the error norms used in finite element analyses, see Appendix I. The class of discretization errors also includes the errors caused by the truncation of iterative procedures and the summation of infinite series. To date discretization error analyses are of **a priori** and **a posteriori** types. This will be discussed further in the succeeding section.

In recent years considerable work has been devoted to the control of round-off errors and its accurate estimation. An overview of the current state-of-the-art is contained in Ref. 21, in which practical methods and algorithms are presented for carrying out basic processes of numerical analysis with guaranteed error bounds. The error bounds account for rounding errors, and wherever appropriate, discretization errors. Some of the algorithms presented in Ref. 21 have been incorporated in the high-accuracy arithmetic package (ACRITH) marketed by IBM (Ref. 22). An important feature of the package is its ability to accumulate inner products exactly. Some flaws in the package have been discussed in Ref. 23. Analytical approaches for developing **a priori** and **a posteriori** bounds for roundoff errors are presented in the two monographs (Refs. 24 and 25), and in the two papers (Refs. 26 and 27). **A priori** and **a posteriori** estimates of roundoff errors, based on linearized perturbation theory, are given in Ref. 28. The primary use of these estimates is to assess the suitability of numerical algorithms based on the sensitivity of the solution to perturbations in the input data. An extensive bibliography on error analysis based on the use of interval mathematics is contained in Ref. 29.

3. ESTIMATION OF DISCRETIZATION ERRORS

Since the development of modern finite element method in the mid 1950's, attempts have been made by engineers and researchers to obtain information about discretization errors, and to prove the convergence of the solutions. Mathematicians became interested in the subject in mid 1960's. Until recently, the two groups worked almost independently, and even now, the emphasis of the work of the two groups is different. The availability of information about discretization errors, in addition to its importance in assessing the reliability of the finite element solutions, provides the possibility of obtaining the "best possible" solution within an allowable cost range.

3.1 Types of Discretization Error Estimates

In general, there are two types of discretization error estimates: a) **a priori** estimates; and b) **a posteriori** estimates.

A priori estimates are based on a knowledge of the characteristics of the solution such as its smoothness, and provide qualitative information about the asymptotic rate of convergence as the number of degrees of freedom goes to infinity. Although **a priori** estimates are accurate for the worst case in a particular class of solutions of the problem (e.g., class of all solutions which have the second derivative bounded by a constant), they usually do not provide information about the actual error for a given finite element model. Nevertheless, **a priori** estimates can be very effective when used in extrapolation techniques. To date **a priori** error estimates (or convergence rate estimates) are available for a wide variety of finite elements. For a brief discussion of **a priori** error estimates see Appendix I. A thorough analysis of these errors is given in many monographs and publications (see, for example, Refs. 30 to 35).

A posteriori estimates use information obtained during the solution process, in addition to some **a priori** assumptions about the solution. For an abstract treatment of the questions related to incomplete information see Ref. 36. Discussion in the succeeding subsections will focus on **a posteriori** estimates which can provide quantitatively accurate measures of the discretization error.

3.2 Discretization Error Diagnostics

Early work by engineers on discretization errors focused on measuring the degree of reliability of the solution obtained with a given finite element model, and developing error diagnostics through checking the satisfaction of: a) the natural boundary conditions (for variational-based finite elements); b) the equilibrium equations by calculating the unbalanced forces (equilibrium defects) both in the interior of the element as well as on its interfaces, see Ref. 37 (for compatible displacement models); and c) the compatibility equations by calculating the strain incompatibility in the interior of the elements as well as across adjacent elements (for equilibrium, mixed and nonconforming models).

The aforementioned error diagnostics are based on considering the finite element solution as an exact solution of the original problem with perturbed input data (e.g., magnitude of unbalanced forces). The main problem is to relate the magnitude of the perturbations to the response quantities of interest (e.g., stresses). Modern **a posteriori** error estimation techniques are often indirectly related to error diagnostics through the effects of superconvergence. Superconvergence, which can occur at special points, refers to extraordinarily high rates of convergence (and concomitant high accuracy) observed at these points.

Equilibrium defects may be deceptive unless properly used and interpreted (see Ref. 38). The proper interpretation of equilibrium defects is still, in general, an open question. For a discussion of some aspects of this subject see Refs. 37, 39 and 40.

3.3 Classical Approaches for Estimating Discretization Errors

The two classical approaches for estimating the discretization errors are:

a) Extension Methods based on comparing the finite element solutions obtained using a sequence of meshes of increasing refinement (h extension); with an increasing degree of the polynomial shape functions (p extension); or using a combination of the two (h-p extension). The computer implementation of the h-, p- and h-p extensions are referred to as the h-, p- and h-p versions, respectively. Extension methods are used to determine the rate of convergence of the finite element solution, and its actual error.

The h-version of the finite element method is the standard one. The mathematical theories for the p- and h-p versions are given in Refs. 41 to 51. Engineering and computational aspects of extension methods are discussed in Refs. 52 to 56. The **a posteriori** error bounds in Refs. 39, 46, 52, 54 and 56 are obtained by using extrapolation lrocedures. The selection of the correct extrapolation procedure is based on the theoretical **a priori** error estimates, and is essential for the effectiveness of the **a posteriori** estimates (see Refs. 39, 46, 52, 54 and 56).

The effectiveness of extension methods in practical applications depends to a great extent on their efficient computer implementation. Herein efficient implementation refers to the combination of the coarse and fine grid solutions in the h- version or the lower- and higher-degree polynomial solutions in the p-version to generate the final solutions. For the h-version efficient computer implementation can be achieved by using multigrid concepts which are widely used in conjunction with the finite difference method (see Ref. 57). For the p- and h-p versions, efficient implementation can be made by using elements with nested set of shape functions (i.e., hierarchic shape functions, see Ref. 52). A brief discussion of the hierarchic shape functions and their computational advantages is given in Appendix II.

b) Dual (or Complementary) Procedure - based on obtaining two solutions using direct and complementary variational methods to provide bounds on the error of the global response characteristics. This approach can be applied to self-adjoint problems formulated using minimum (rather than stationary) variational principles. For elasticity problems this is accomplished by using the direct and dual formulations of the problem (based on the principles of the minimum potential energy and the minimum complementary energy) to obtain upper and lower bounds of the error measured in the energy norm (see Ref. 58). Applications of the dual procedure are given in Refs. 59, 60 and 61. The procedure is expensive but leads to guaranteed upper bounds of the error (measured in the energy norm) which are not pessimistic.

3.4 A Posteriori Error Estimates and Error Indicators

In recent years considerable effort has been devoted to the development of computable (i.e., element by element) **a posteriori** error estimates based on local error estimators. These error estimators are based on information obtained during the solution process itself. The work presented in Refs. 62 to 66 started this activity and was extended in various directions (see Refs. 67 to 77). For example, in Ref. 67 error estimates are presented for linear elements (with linear shape functions) using **a priori** bounds on single elements as well as **a posteriori** information about the second derivatives. All the error estimates developed in the cited references are for compatible (or conforming) finite element models.

The finite element solutions have equilibrium defects in the interior and on the portion of the boundary where tractions are prescribed; as well as jumps in the tractions at interelement boundaries. The equilibrium defects are nearly self-equilibrating. The square of the energy norm of the total error (i.e., the strain energy of the error) can be obtained by summing the squares of the energy norms of the errors for individual elements, each obtained by solving (or estimating the solution of) an auxiliary problem characterizing the equilibrium defects in that element (see Refs. 62, 69, 73 and 77). In case of odd-degree elements (elements with odd-degree polynomial shape functions), the boundary defects dominate the residual, and for even-degree elements the interior defects are dominant (see Ref. 78). The energy norms of the aforementioned auxiliary problem can be accurately estimated or obtained by a finite element with properly selected shape functions. Hierarchic shape functions are particularly useful for this purpose (see Refs. 68, 71, 79 and 80).

The estimation of the square of the energy norm of the total error through summation of elemental contributions can also be applied to nonlinear problems (including limit and bifurcation points in the solution space), see Refs. 62, 64, 79, 81 and 82.

The local error estimators (computed for individual elements) are referred to in Ref. 66 as **error indicators** $\eta\ (\Omega^{(e)})$, where $\Omega^{(e)}$ is the element domain. Some approaches for constructing error indicators are described in Refs. 79 and 83. Error indicators can be used to construct overall problem error estimators, ε, for the problem. The expression of ε in terms of η depends on the particular error norm being used (see Refs. 66, 67, 71, 73, 76, 83, 84 and 85). For example, the energy norm error is given by:

$$\varepsilon^2 = \sum_{\text{elements}} \eta^2\ (\Omega^{(e)})$$

Other error estimators proposed include residual errors in the constitutive relations (Refs. 40, 74 and 86); and in the strain-displacement relations (Ref. 87). Error estimators for singular perturbation problems are given in Refs. 83 and 88.

3.5 Reliability of Error Estimators

The **a posteriori** error estimators described in the preceding subsections are asymptotically correct if the ratio of the estimator to the true error converges to one as the true error tends to zero (when the mesh size tends to zero, or when the degree of polynomial shape function goes to infinity). Therefore, in order to assess the reliability of these estimators for a finite grid, an effectivity index, θ, was introduced in Refs. 65 and

89. The effectivity index is defined as follows:

$$\theta = \frac{\varepsilon}{\|e\|}$$

where ε is the **a posteriori** error estimator and $\|e\|$ is the norm of the error. The estimator is asymptotically correct when $\theta \to 1$ as $\|e\| \to 0$.

For practical applications it is essential that $|\theta - 1|$ be small (e.g., less than 0.2 when the error $\|e\|$ is of the order of 10% of the norm of the solution, and should decrease as the error decreases, see Ref. 62).

It is preferable that θ be greater than one so that the true error is overestimated. The asymptotic correctness of the error estimator is related to the superconvergence effects. In addition, some assumptions about the meshes and the solution are needed, without which the quality of the estimators can be low. The assumptions about the meshes are satisfied when they are constructed adaptively. In such cases $|\theta - 1|$ is of the order of 0.1. The mathematical theory of error estimators is discussed in Refs. 73, 77, 78 and 85. Application to one-dimensional problems are given in Refs. 50 and 63.

3.6 Error Estimators for Stationary and Transient Problems

Most of the error estimates reported in the literature are for stationary problems (linear and nonlinear). Error estimates have also been developed for some transient problems (see Refs. 79, 83 and 90 to 97). Stationary problems include static stress analysis and are mostly governed by elliptic systems of differential equations. The transient problems include dynamic response and wave propagation problems and are governed by equations of the parabolic and hyperbolic types.

Among the elliptic structural mechanics problems for which **a posteriori** error estimates exist are one-dimensional rod with variable stiffness, second-order partial differential equations, and two-dimensional elasticity. Numerical results have been presented for plane stress problem of a cracked plate, plane strain problem of a dam, L-shaped region with a corner singularity, see Refs. 62, 68-70, 84-86 and 88. The cited references contain numerical examples that demonstrate the effectiveness of the error estimators. Most of the problems presented in the cited references are governed by second-order differential equations and are modeled by using conforming elements. Error estimates for plate bending problems and for the boundary element method are discussed in Ref. 98. Only few finite element programs have facilities for error estimation, based on the computation of error indicators (see, for example, Refs. 81, 99 and 100). This will be discussed further in Section 4.3.

3.7 Comments on Error Estimators

1. The selection of an error measure depends on the goal of the computation, and on which response quantities need to be accurately determined, e.g., buckling loads, stresses in a critical zone; stress intensity factors at crack tips; or density in compressible flow problems (for a discussion of the goals of stress analysis see Ref. 101). This is particularly important since the smallness of one measure does not guarantee the smallness of the errors in other response quantities, computed from the finite element solution. An example of this was given in Ref. 39 where a small energy norm error was associated with large errors in the computed stresses.

2. For the elliptic problems considered in the literature the computation of the error indicators takes a small percentage of the total solution time. Significant differences can occur in the cost of computing different error estimators. Also, error estimators for higher-order elements are more expensive to compute than for lower-order elements.

3. Pointwise estimation of the errors in detailed response quantities (e.g., stresses, displacements and stress intensity factors) is considerably more difficult than the evaluation of the local energy norm errors. For simple problems (e.g., Saint-Venant torsion problem, one-dimensional problems, and two-dimensional plane elasticity problems), attempts have been made to obtain pointwise error estimates for stresses. These error estimates are based on a postprocessing approach (Ref. 102) or on heuristic arguments using the experience gained from finite difference discretizations (Refs. 72 and 103).

4. When the goal of the analysis is the accurate determination of detailed response quantities such as the stresses, pointwise error estimates are needed. If these pointwise estimates are not available, then energy norm errors are computed, and additional numerical tests are performed such as element-by-element equilibrium test and action-reaction test (Ref. 104).

4. THE ADAPTIVE IMPROVEMENT OF FINITE ELEMENT SOLUTIONS

Adaptive improvement of finite element solutions refers to improving the quality of the solutions by enriching the approximation in some manner so as to achieve the "best" solution for a given computational effort (or cost). Adaptive procedures have in common with feedback approaches the fact that the computed information is used for steering the process. However, it is the attempt to obtain the "best" solution that distinguishes adaptive processes from feedback approaches. Therefore, an adaptive process is a feedback approach which is optimal with respect to clearly defined objectives (see Refs. 89, 105 and 106). The adaptive process is normally performed after an initial solution is already available; and regions of the solution domain where the accuracy is not satisfactory have been identified according to a preselected set of criteria. This process can be iterated using the last solution as the new initial solution. The term adaptive refers to the fact that each model modification step uses, in some optimal way, the information provided by previous steps. For a discussion of the notion of adaptivity in numerical procedures see Refs. 89 and 105.

Adaptive processes can result in considerable savings in computational effort if an initial coarse grid is used followed by an effective strategy to improve the quality of the solution wherever needed. In most of the literature on feedback approaches, neither the goals of the process nor the criteria used to test whether the feedback approach is adaptive, are defined. A feedback approach can be adaptive (and hence optimal) with respect to a set of criteria and nonadaptive with respect to other criteria. In the h-version of the finite element method, the feedback consists of the construction of successive meshes to achieve the goal of computation, and in the h-p version both the shape functions and the mesh are enriched simultaneously.

Mesh selection and improvement can be carried out by using AI-based expert systems, or a combination of expert systems and a feedback approach. The expert systems act as an expert consultant which decides the direction to steer the adaptive process (Ref. 107), or can be used as a preprocessor for designing the finite element model.

In general, a feedback process has a number of key elements including (see Refs. 76, 84, 88, 89 and 105):

a) **error measure.** Each model modification step should aim to minimize some error measure. This error measure is related to the goal of the computation;

b) **the feedback procedure** (or adaptive modification strategy) which aims at minimizing the error measure selected in a) in the most effective manner; and,

c) **stopping criterion.** This criterion is related to the goal of the computation, usually through reducing the error measure to a prescribed tolerance.

The efficiency of the feedback process is measured through a well-defined cost function (see Ref. 106). If either the error measure or the cost function is changed, the feedback algorithm and the resulting finite element model may change significantly. As an example to this, the use of the error measures based on the energy norm and the maximum stress norm can lead to two quite different finite element models.

4.1 Strategies for Adaptive Improvement of Solutions (Feedback Procedures)

Among the different strategies used for adaptive improvement of the finite element solutions (feedback procedures) are the following four:

a) successive selection of the meshes;

b) moving the nodes (node relocation);

c) successive selection of the local order of the approximation; and,

d) simultaneous selection of the meshes and the local order of approximation.

In the **first strategy,** the degree of the elements is kept constant and the mesh is constructed in a feedback manner. It is usually referred to as the h-method. It can result in refining or de-refining the mesh. De-refinement of the mesh may be necessary in transient (time-dependent) problems and in nonlinear problems when continuation method is used for their solution. A theoretical analysis of the various feedback approaches and their optimality in one-dimensional problems is given in Refs. 80 and 108. The feedback approaches for multidimensional problems are based on heuristic arguments and analogies with one-dimensional problems. Note that the feedback algorithms involve a direct or indirect use of the error indicators discussed in Section 3.4; and the adaptive strategies usually aim at making the error indicators for different elements nearly equal.

In the **second strategy,** the quality of the finite element solution is improved by optimizing the location of the nodes keeping the number of degrees of freedom fixed and the degree of the elements constant. This is usually referred to as R-method (Refs. 109 and 110). For hyperbolic equations the R-method is more often used in conjunction with finite-difference methods than with finite element methods. Because of the kinship between the two methods, the basic ideas of adaptive approaches used in conjunction with finite differences can be applied to the finite element methods. For a discussion of adaptive finite-difference methods see, for example, Refs. 111 to 116.

In the application of the moving element method to initial value (transient) problems, a fixed number of finite elements are moved so as to concentrate the computational grid in regions containing nonuniformities in the solution (e.g., shocks, near shocks, boundary layers, and sharp moving fronts). Both the nodal amplitudes and nodal positions move continuously with time in such a way as to satisfy simultaneous ordinary differential equations (in time) which minimize the partial differential equation residuals. Galerkin-type finite elements are typically used for the spatial discretization and stiff ordinary differential equation solvers are used for the temporal integration. Several mesh moving techniques have been developed for one-dimensional parabolic and hyperbolic systems (see, for example, Refs. 95, 97, and 117 to 122). For two-dimensional systems, substantial difficulties still remain including mesh tangling and gross distortion (see, for example, Refs. 79, 94 and 123 to 125).

The **third strategy** is based on successive selection of the order of the polynomial approximation inside the elements. It is usually referred to as the p-method and has the advantages over the first two methods of being easy to implement and of providing a simple formula for the error indicator. This is particularly true when hierarchic basis functions are used, since the stiffness matrices and load vectors corresponding to a polynomial of degree p are embedded in the stiffness matrices and load vectors corresponding to polynomials of degrees $\geq$ p+1 (see Refs. 79, 83 and 126 to 128). Application of this strategy to the solution of three-dimensional thermoelastic problems of anisotropic solids is given in Ref. 129.

The **fourth strategy,** based on simultaneous selection of the meshes and the local order of approximation is referred to the h-p method. The mathematical foundations for the adaptive h-p method and R-methods are much less developed than for the other methods.

A variety of feedback approaches are now available for differential equations of different types: elliptic, parabolic, and hyperbolic; linear and nonlinear; describing a broad spectrum of engineering problems. Significant contributions were made in the last few years (see, for example, Refs. 67 to 71, 77, 79, 83, 84, 90 to 92, 94 to 97, 116 to 121, 123, 125 and 130 to 132).

Some of the feedback approaches are based on older ideas such as the local extrapolation techniques and the **deferred correction approach.** The deferred correction approach is used in various software packages for solution of initial and two-point boundary value problems. It is based on using the numerical solution obtained to construct a pseudo or neighboring problem whose exact solution is known (e.g., polynomial or spline interpolation of the discrete numerical solution). The pseudo problem is then solved using the same finite element model as that used for the original problem. The error in the pseudo problem is assumed to be a close approximation of the error in the original problem and is used as a correction to that solution. The technique has been successfully applied to the numerical solution of stiff systems of ordinary differential equations (Refs. 133 and 134) and has recently been extended to finite element boundary-value problems (Refs. 135 to 137).

4.2 Optimal Finite Element Mesh

An optimal finite element mesh is the one, among the class of admissible meshes, for which the distribution of the nodes (and associated degrees of freedom) minimize the error (measured in a particular norm). In the class of admissible meshes, among other things, the local order of approximation is specified. Obviously, changing the error norm or the class of admissible meshes may change significantly the optimal mesh.

The first attempts to produce optimal meshes for variational-based finite elements were based on considering the nodal positions to be unknowns, and minimizing the functional (potential energy) with respect to both the nodal degrees of freedom and the nodal positions (Refs. 138 to 141). Such an approach was found to be impractical since it results in nonlinear equations, even for linear problems.

The construction of an optimal mesh, which minimizes the error, results in an ill-posed problem. A small movement of a node of the optimal mesh leads to a second-order change in the error. To alleviate this problem **nearly optimal** meshes are considered. For transient problems the feedback procedure used for generating nearly optimal meshes, is based on using various spring constants and other stabilization parameters in the finite element model (see Refs. 94 and 117).

More recently, it was proposed to characterize an **optimal mesh** as one for which the error indicators for different elements are nearly equal. For one-dimensional problems it was shown that the mesh with equal error indicators is asymptotically optimal in the sense that it leads to the minimum error for a given number of degrees of freedom (see Refs. 51

and 65). Most of the currently-used adaptive strategies attempt to construct nearly optimal meshes (with nearly equal error indicators). This is referred to as the **equilibration principle** for mesh construction (Ref. 66). Important applications of this principle for the cases when the solution has singularities are given in Refs. 53, 142 and 143. A detailed analysis of the optimal meshes and distribution of element degrees for a one-dimensional problem is given in Ref. 51. In the cited reference the characteristics of the optimal meshes, as well as the upper and lower bounds of the error are given, for both the h- and h-p methods.

The mathematical foundations for the construction of optimal meshes is much better developed for elliptic problems than for parabolic and hyperbolic problems. For initial value (transient) problems the reader is referred to Refs. 79, 83, 90 to 92, 125 and 131. In principle, if the mesh constructed in the feedback mode has nearly the same accuracy (measured in some norm) as that of the optimal mesh, then the feedback process is defined to be optimal. Hence, the lower estimates for the error of an optimal mesh have close relation to the construction of an adaptive approach.

4.3 Computer Implementation of Feedback Procedures (Adaptive Strategies)

One of the most difficult aspects of feedback procedures is the computer implementation of the automated enrichment or adaptive algorithm. This includes the selection of the data structure used for the representation (and refinement) of the grid, and grid management. The implementations of feedback processes are different for elliptic, parabolic and hyperbolic problems (see Refs. 75, 81, 93, 96, 99, 100, 144 to 147). Most of the aforementioned papers on adaptive improvement of numerical solutions give numerical examples to demonstrate the proposed strategies, but do not present general computer programs for implementing these strategies.

To the authors' knowledge none of the large-scale, general-purpose commercial finite element systems has facilities for adaptive enrichment of solutions. However, some special-purpose and pilot finite element systems have implemented adaptive algorithms for practical performance studies. Among these systems are the Finite Element Adaptive Research Solver (FEARS) developed at the University of Maryland in 1979-1980 (Ref. 99); the EXPDES system developed in Belgium (Ref. 100); the PLTMG system at the University of California, San Diego (Ref. 81); the self-adaptive finite element simulator (SAFES code) developed at the University of Wyoming (Ref. 148); and the PROBE system developed by NOETIC Technologies Corporation in St. Louis (Ref. 52). The first two systems and the last system are designed to solve two-dimensional elliptic problems. The three systems (FEARS, PLTMG and EXPDES) are based on the h-version. The last system (PROBE) is a commercial program which uses the p- and h-p extensions. Although it is currently not self-adaptive, the developer is adding the automatic enrichment capability to it. The SAFES code performs the spatial refinement for elliptic and parabolic linear systems. All of the aforementioned programs were developed for sequential computers. It is anticipated that modern parallel architecture will have a strong impact on the implementation of adaptive methods (see Refs. 145 and 149).

4.4 Comments on Adaptive Strategies and Feedback Procedures

The following comments can be made concerning the adaptive strategies reported in the literature:

a) Adaptive processes have been used extensively in the solution of ordinary differential equations (initial value and two-point boundary value problems). Modern program packages incorporate feedback approaches.

b) Detailed rigorous mathematical analyses of adaptive algorithms are only available for some one-dimensional problems. For multidimensional problems the algorithms used for the adaptive construction of the desired sequence of nearly equilibrated meshes (with nearly uniform error indicators) are largely of a heuristic nature.

c) The efficient computer implementation of adaptive processes has not matured yet. However, based on the experience acquired so far certain directions are evolving.

For example, in the adaptive h-version the use of a tree structure for the data was found to provide the most efficient data management system. In the adaptive p- and h-p versions the use of hierarchic shape functions is optimal because the stiffness and load matrices for a certain level of refinement remain unchanged when a higher level is introduced, and iterative methods can be efficiently used for solving the algebraic equations associated with different levels of refinement (similar to the multigrid finite difference setting); see, for example, Refs. 57, 75, 81, 96, 99, 100, 144 and 147.

d) For elliptic problems adaptive strategies based on mesh enrichment (h-, p- and h-p methods) are more commonly used than those based on node relocation (R-method). For a large class of engineering problems, the h-p extension has an exponential rate of convergence with respect to the number of degrees of freedom, while the h- and p-extensions have only a polynomial rate of convergence (see Appendix I). However, if properly-graded meshes are used, the performance of the p-extension in the pre-asymptotic range is very close to the best performance attainable with the h-p extension. Moreover, for quasi-uniform meshes the rate of convergence of the p-extension cannot be worse than that of the corresponding h-extension (based on the same total number of degrees of freedom). In the presence of singularities, the rate of convergence of the p-extension can be twice that of the h-extension (Refs. 44 and 46 to 48).

e) For initial value (parabolic and hyperbolic) problems moving meshes have to be used either continually or by adaptively determined intervals (Ref. 92). The major drawback of moving mesh techniques is that they do not add and/or delete elements as the computation progresses, and are generally not capable of generating a solution with a prescribed level of accuracy. To remedy this drawback a strategy combining mesh moving and refinement (or de-refinement) has recently been proposed (see Refs. 115, 124 and 146).

f) The p-version results in more dense matrices than the h-version and is more convenient and simpler to implement. In addition, the treatment of curved elements require special care when higher-degree polynomials are used (viz., smooth mapping such as that provided by the blending function method - see Appendix II).

g) The p-version was found to be not too sensitive to the "locking" problems encountered in thin plate and curved shell structures (see Ref. 150).

5. POSTPROCESSING OF FINITE ELEMENT SOLUTIONS TO IMPROVE THEIR QUALITY

Often the primary objective of the finite element analysis is to determine certain response quantities, such as stress intensity factor, stresses, strains, displacements, or flux in some area of the domain (or structure). The simplest approach is to compute this data directly from the finite element solution. For example, stresses are calculated by differentiating the displacement field and using the constitutive relations. This results in lower accuracy for the stresses (and strains) than for the displacements. Recent studies have demonstrated that the accuracy and rate of convergence of stresses (and strains) depend on how (and where) they are computed. The number of publications on postprocessing approaches, based on the superconvegence concepts (i.e., increased accuracy and improved rates of convergence), is steadily increasing (for example, Ref. 151 lists 200 publications on superconvergence). Most of the publications deal with elliptic problems. Among the proposed postprocessing approaches for elliptic problems are:

a) evaluating the stresses at numerical quadrature points and determining their values at the displacement nodes by extrapolation;

b) averaging or smoothing based on projection techniques;

c) using the extraction techniques;

d) computing the stresses using the discarded structural equations (corresponding to prescribed displacement boundary conditions); and,

e) using iterative refinement techniques which build continuous stress field and displacement field from the classical solution.

The first approach is by far the most commonly used. Superconvergence has been observed for displacement derivatives and stresses evaluated at the quadrature points of uniform meshes (using Gaussian quadrature formula of the proper degree- see Refs. 152 to 159).

Averaging and smoothing techniques can be very effective for postprocessing of the finite element solution. Several approaches for averaging and smoothing have been proposed in the literature (see, for example, Refs. 160 and 161). The analysis of simple averaging techniques is discussed in Refs. 151 and 162. The superconvergence for general (nonuniform) meshes seems to be an open question (see Ref. 163).

Extraction techniques are based on using analytic expressions of the function which approximate the kernel of the functional of interest. Because of its higher cost of implementation, this approach is feasible for computing the stress intensity factor, or for obtaining accurate stresses only in a small area (critical zone) of the structure (see Refs. 52, 102 and 164 to 169). Extraction methods for computing the stress intensity factor in problems with singularities at the corner and along the edges are presented in Ref. 170.

The fourth approach has recently been extended to the computation of stresses at the interfaces between elements using the previously computed stiffness matrices and load vectors (Ref. 171). For one-dimensional problems, the technique requires little computation and its implementation is straightforward. However, for two-dimensional problems, the stress calculation involves the evaluation of contour integrals and the implementation is more complex. Moreover, new difficulties arise from the presence of corners in polygonal domains.

Iterative refinement techniques, such as those reported in Refs. 172 to 174, have been quite effective in reducing equilibrium errors and significantly improving the accuracy of the calculated stresses. These techniques were recently shown to be equivalent to mixed formulations in which the stresses (or strains) and displacements are used as primary variables (Ref. 175). The techniques have high potential for nonlinear problems since the iterative improvement can be done simultaneously with those necessitated by the nonlinearity, and no additional computational cost is involved.

The aforementioned approaches were discussed in the context of elliptic problems. Less work has been done on superconvergence concepts and postprocessing approaches for parabolic problems, and much less for hyperbolic problems (see Refs. 176 and 177).

6. FUTURE DIRECTIONS FOR RESEARCH

Considerable attention has been devoted to the estimation and control of discretization errors which is manifested by the development of a mathematical theory for **a posteriori** error estimates and feedback approaches (adaptive refinement strategies). However, the theory is very incomplete and has been numerically tested only on a simple set of problems. Moreover, adaptive strategies have only been incorporated into special-purpose and pilot finite element systems used for practical performance studies. To-date none of the general-purpose commercial finite element systems have facilities for estimating the error in the finite element solution, or for adaptive improvement of this solution. To remedy this situation major advances are needed in the theory, strategies and algorithms for implementation of error estimation and control. Some of these advances are listed subsequently.

1. Development of reliable measures for estimating the errors in the finite element predictions of the major response quantities (selected by the analyst according to the goal of computation). These errors are due to the simplifying assumptions made in abstracting the mathematical model from the real system (structure); uncertainties in the input information (of the mathematical model); and numerical discretization of the continuous mathematical model. The following observations can be made about estimating the different types of errors:

a) The use of hierarchic modeling strategies and formulations (e.g., hierarchy of two-dimensional plate and shell theories) allows the estimation of modeling errors as well as the adaptive refinement of the model whenever needed.

b) The use of stochastic formulations allows the study of the effect of uncertainties in input information on the response predictions of the finite element model.

c) Practical **a posteriori** error estimators should satisfy the following four criteria.

i) provide reliable local assessment of the error;

ii) are computationally inexpensive to evaluate;

iii) applicable to a wide class of discrete finite elements; and,

iv) easy to use in conjunction with adaptive improvement.

Note that the choice of the error estimator depends on the goal of the computation (e.g., buckling analysis, detailed stress analysis, determination of stress intensity factor, etc.).

In the absence of practical error estimators, attempt should be made to develop computationally inexpensive **error sensors** or correction indicators which give an indication of the regions where mesh refinement is needed. The sensitivity derivatives of the solution to the degree of approximation can be used as error sensors. Also, numerical checks to validate the solution should be conducted. Examples of these checks are element-by-element equilibrium check and action-reaction test.

2. Development of efficient adaptive improvement strategies. In particular, the development of finite element models based on the p-hierarchic shape functions for higher-order problems (e.g., shear-flexible plates and shells) along with efficient numerical algorithms for reanalysis. The element computation should include the esstential information needed for **a posteriori** error estimation.

3. Efficient computer implementation of adaptive strategies. This includes using novel computer science concepts for data management.

4. Development of adaptive strategies for new computing systems. A number of attempts have been made to develop parallel numerical algorithms and parallel software for adaptive processes (see, for example, Refs. 145 and 149). However, the effective use of parallelism in adaptive strategies requires the use of:

a) primitive-variable formulations (e.g., three field mixed formulation in structural mechanics);

b) domain decomposition or substructuring (with minimization of interfaces); and

c) operator splitting techniques to uncouple the resulting algebraic equations.

5. A systematic assessment of the postprocessing techniques used for improving the accuracy of derivative calculation (stresses and strains in displacement finite element models) is needed. Also, the possibility of using these techniques for error sensing should be investigated.

6. Development of AI-knowledge based system for efficient use of finite element programs by the nonexperts.

7. Selection of meaningful benchmark problems which have the essential features of practical problems to test the theory and the effectiveness of the adaptive strategies developed.

7. CONCLUDING REMARKS

Status and some recent developments in the techniques for error estimation and adaptive improvement of finite element solutions are summarized. Discussion focuses on a number of aspects including: the major types of errors in finite element solutions; techniques used for **a posteriori** error estimation and the reliability of these estimators; the feedback and adaptive strategies used for improving the finite element solutions; and postprocessing approaches used for improving the accuracy of stresses and other important engineering data.

The status of finite element modeling, error estimation and adaptive improvement of finite element solutions can be summarized in the following:

1. **Mathematical modeling errors.** Despite their importance, have not received enough attention in the literature. To date no simple and general approach is available for assessing these errors.

2. **Selection of the formulation and initial finite element models.** These are largely based on intuition and experience gained from similar problems. The modeling process can be aided by the information available on the asymptotic rates of convergence (**a priori** estimates of the discretization errors) for different finite element models based on

different formulations. These **a priori** error estimates predict the asymptotic rate of convergence as the mesh size tends to zero (or the degree of polynomial shape function goes to infinity). However, except in the adaptive improvement of finite element solutions, **a priori** error estimates are currently not used in constructing the initial finite element models. The knowledge-based systems, in spite of their high potential have not been sufficiently developed to aid in the modeling process.

3. **Assessment of the reliability of the finite element solution.** This includes:

a) Selection of an appropriate measure for the discretization error, which depends on the goal of the computation, and on which response quantities need to be accurately determined; and,

b) Calculation of **a posteriori** estimates for the discretization error using locally computable (element-by-element) error indicators.

The most commonly used error measures are the interior and boundary residuals; and the local energy norm error. The first represents the equilibrium defects in the interior; on the portion of the boundary where tractions are prescribed, as well as the jumps in the tractions at interelement boundaries. The energy norm error is defined as the square root of the strain energy of the error. Pointwise error estimates for detailed response quantities (e.g., stresses and displacements) are available only for simple problems (e.g., Saint Venant torsion problem and one-dimensional problems).

Most of the **a posteriori** error estimates reported in the literature are for stationary, (elliptic) problems (e.g., plane stress analysis of structures subjected to conservative loading and modeled by conforming displacement finite elements). No a posteriori error estimates are available for nonconforming elements.

4. **Postprocessing of finite element solutions.** The accuracy of strains, stresses and stress intensity factors computed from compatible finite element models can be improved by postprocessing. Several approaches have been proposed in the literature. A systematic and detailed assessment of these approaches is needed.

5. **Quality control for finite element solution.** This is accomplished by using adaptive strategies which are feedback processes aimed at achieving the desired accuracy with the least computational effort and/or cost. The adaptive process is performed after an initial solution is already available and the regions of the solution domain where accuracy is not satisfactory have been identified (according to the preselected error measure). Four adaptive strategies have been proposed in the literature. These include: i) successive selection of the meshes (h-method); ii) moving the nodes (R-method); iii) successive selection of the order of the polynomial approximation inside some elements (p-method); and iv) simultaneous selection of the meshes and of the local order of approximation (h-p method).

For stationary (elliptic) problems the h-, p- and h-p methods are more commonly used than the R-method. For time-dependent (parabolic and hyperbolic) problems adaptive strategies are based on either the moving mesh techniques or combination of mesh moving and refinement (or de-refinement).

The mathematical theory for **a posteriori** error estimates and feedback approaches is very incomplete and has been numerically tested only on a simple set of problems. In spite of the considerable attention devoted by engineers and mathematicians to the subject of error estimation and control, none of the large-scale commercial finite element systems has facilities for error estimation or adaptive improvement.

The maturation of the technology of estimation and control of discretization errors, and the incorporation of this technology into general-purpose finite element systems, will allow the analyst to select only: a) the initial discrete model which is sufficient to resolve the topology of the structure (or the geometry of the domain); and b) the error measure and the tolerance. Then the finite element system can automatically refine the model until the selected error measure falls below the prescribed tolerance. The strategy for adaptive improvement can either be specified by the user or automatically selected by the program (possibly with the aid of an AI-based expert system) in such a manner as to minimize the cost of the analysis. The research areas that are needed to make error estimation and adaptive improvement practical are identified in the paper.

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APPENDIX I - ERROR NORMS

Error norms are introduced to provide quantitative measures of the magnitude of the error in the finite element solutions. These are measures of the magnitude of the discretization errors in certain quantities, and therefore, their choice depends on the goal of the analysis. Three principal choices of error norms are commonly used in finite element methods: the energy norm $\| e \|_E$, the mean-square norm $\| e \|_0$, and the maximum or infinity norm $\| e \|_\infty$.

The energy norm is defined as the square root of the energy of the error, i.e.

$$\| e \|_E = \left(\int_\Omega U(e) \; d\Omega \right)^{1/2} \tag{A.1}$$

where Ω represents the solution domain; U represents the strain energy; e is the discretization error function (difference between the exact and the finite element solutions, respectively).

The mean-square (or L^2) norm measures the root-mean-square of the errors over the solution domain, and is defined by:

$$\| e \|_0 = \left(\int_\Omega e^2 \; d\Omega \right)^{1/2} \tag{A.2}$$

The maximum norm measures the maximum absolute value of the error over its domain:

$$\| e \|_\infty = \max_{x \in \Omega} \; |e(x)| \tag{A.3}$$

In Eqs. (A.2) and (A.3), e refers to displacement, stress or other response quantity. Of the three error norms, the energy norm is the most commonly used. It provides a meaningful quantitative measure for the error in the finite element approximation and is equivalent to the root-mean-square of error in stresses. For practical engineering applications the maximum norm is not convenient to use.

For conforming elements based on minimization principles the strain energy of the error decreases monotonically with systematic changes of the discretization (refining the mesh or increasing the degree of the polynomial approximation).

For a wide variety of finite elements **a priori** error estimates have been developed. These estimates predict the asymptotic rate of convergence as the mesh size tends to zero (or the degree of polynomial interpolation function goes to infinity).

A priori estimates for stationary (elliptic) and time-dependent (parabolic and hyperbolic) problems are given in the literature on mathematical theory of finite elements. Both conforming and nonconforming elements; single-field and multifield models have been studied.

For simplicity, only linear elliptic, self-adjoint, positive-definite problems (example, linear static stress analysis problems modeled by compatible displacement models) are considered herein. For these problems the discretization error consists of two distinct parts: a) approximation error; and b) perturbation error. The first (approximation) error characterizes the approximation of the finite element solution and depends on:

a) Order of the highest-order derivative used in the formulation;

b) Degree of the elements;

c) Dimension of the largest finite element (e.g., diameter of the element); and

d) Smoothness of the solution.

For uniform meshes when the solution is not smooth (e.g., due to the presence of singularities in the solution inside the region or on the boundary) the convergence rate decreases.

Perturbation errors result from geometric approximation of the boundary; approximation of the essential boundary and continuity conditions; and numerical approximation of integrals. The finite element solution is said to be optimal if the perturbation errors do not exceed the approximation errors.

For elliptic problems in which the perturbation error is small, the error estimate can be given in the following simple form:

$$\| e \| \leq Cf(N) \tag{A.4}$$

where C is a positive constant depending upon the data of the problem; N is the number of degrees of freedom in the model; and f(N) is a function of N; which depends also on the choice of interpolation functions; the element size; and the smoothness of solutions.

If $\| e \|$ is selected to be the energy norm $\| e \|_E$, the function f(N) can be given by:

$$f(N) = N^{-\beta} \tag{A.5}$$

for the h and p extensions (Refs. 47 and 48); and

$$f(N) = 1/e^{\gamma N^{\theta}} \tag{A.6}$$

for the h-p extension (Ref. 46).

where β, γ and θ are positive constants which can be determined by using **a priori** error estimates.

For the h-version the asymptotic rate of convergence, β, is the absolute value of the slope of the plot log $\| e \|_E$ versus log N. A large β signifies a rapid decrease in the error with increasing N. For the h-p version the asymptotic rate of convergence can be conveniently obtained by plotting log $\| e \|$ versus N^{θ}.

APPENDIX II - HIERARCHIC BASIS FUNCTIONS

Hierarchic basis functions refer to nested sets of shape functions which have the property that each set is explicitly contained in the succeeding sets. The nesting of the shape functions is similar to that of the terms of the Fourier series expansion, and is usually used in conjunction with the p-extension. However, it can be interpreted in the context of the h-extension when the shape functions associated with a coarse grid are explicitly contained in successively refined grids. The nesting of the shape functions results in the nesting of the elemental stiffness matrix and load vector (i.e., the stiffness matrix and load vector associated with the lower-degree shape functions are explicitly contained in successively higher-order approximations). In the p-extension, the lower-degree shape functions are contained in successively higher-degree basis. These basis functions are constructed from integrals of Legendre polynomials and are referred to as p-hierarchic shape functions.

For one dimensional elements and two-dimensional quadrilateral master elements with C^0 continuity the p-hierarchic shape functions are listed subsequently (see Ref. 52).

B.1 One-dimensional Elements

In one dimension the set of p-hierarchic shape functions are:

$$N_1 = 1/2\ (1-\xi)$$

$$N_2 = 1/2\ (1+\xi)$$

$$N_i = \sqrt{\frac{2i-3}{2}} \int_{-1}^{\xi} P_{i-2}(t)dt \quad , \quad i \geq 3$$

where $P_i(t)$ is Legendre polynomial of degree i; ξ is a dimensionless coordinate, $-1 \leq \xi \leq 1$.

The shape functions N_i have the following orthogonality property

$$\int_{-1}^{+1} \frac{\partial N_i}{\partial \xi} \frac{\partial N_j}{\partial \xi} \, d\xi = \delta_{ij} \qquad i,j \geq 3$$

where δ_{ij} is the Kronecker delta.

Note that for $i \geq 3$, $N_i(-1) = N_i(+1) = 0$.

B.2 Two-Dimensional Quadrilateral Elements

These have three kinds of hierarchic shape functions:

a) Nodal shape functions. These are the 4 standard shape functions for a quadrilateral element:

$$\bar{N}_1 = 1/4 \ (1-\xi)(1-\eta)$$

$$\bar{N}_2 = 1/4 \ (1+\xi)(1-\eta)$$

$$\bar{N}_3 = 1/4 \ (1+\xi)(1+\eta)$$

$$\bar{N}_4 = 1/4 \ (1-\xi)(1+\eta)$$

b) Side Shape Functions. There are 4 (p-1) shape functions associated with the sides of elements ($p \geq 2$). The shape functions associated with the sides $\eta=-1$ and $\xi=+1$ have the form:

$$\bar{N}_i^{(1)} = 1/2 \ (1-\eta) \ N_i(\xi)$$

$$\bar{N}_i^{(2)} = 1/2 \ (1+\xi) \ N_i(\eta) \qquad (i = 3, \ \ldots \ p)$$

with similar definitions for the shape functions $\bar{N}_i^{(3)}$ and $\bar{N}_i^{(4)}$ associated with the sides $\eta=+1$ and $\xi=-1$; and $N_i(\xi)$ and $N_i(\eta)$; $i \geq 3$, are the one-dimensional shape functions defined in the preceding subsection.

c) Internal Modes. There are 1/2(p-2)(p-3) internal modes ($p \geq 4$). The first mode

$$\bar{N}_i^{(0)} \ (\xi,\eta) = (1-\xi^2)(1-\eta^2)$$

The other internal modes are obtained by multiplying $N_1^{(0)}$ by Legendre polynomials and products of Legendre polynomials, as follows:

$$\bar{N}_2^{(0)}(\xi,\eta) = \bar{N}_1^{(0)} P_1(\xi)$$

$$\bar{N}_3^{(0)}(\xi,\eta) = \bar{N}_1^{(0)} P_1(\eta)$$

$$\bar{N}_4^{(0)}(\xi,\eta) = \bar{N}_1^{(0)} P_2(\xi)$$

$$\bar{N}_5^{(0)}(\xi,\eta) = \bar{N}_1^{(0)} P_1(\xi)\, P_1(\eta)$$

$$\vdots$$

with similar definitions for the other shape functions.

B.3 Two-Dimensional Curved Sided Elements

The implementation of proper mapping procedures is important in order to ensure that the boundary approximation will not degrade the performance of the element. One approach to accomplish the mapping is through the use of linear blending function method (see Refs. 178 and 179), in which the Cartesian coordinates x,y are expressed as functions of the local element coordinates ξ and η .

The inverse mapping can be obtained by using, for example, Newton-Raphson technique. Isoparametric mapping can be viewed as a special application of the blending function method in which the sides are represented by polynomial functions.

B.4 Comments on Hierarchic Shape Functions

The following observations can be made about hierarchic shape functions.

1. The quality of the finite element approximation depends on the selected finite element mesh and the degree of polynomial interpolation function. The use of hierarchic and nonhierarchic shape functions, of the same degree, results in the same approximation. In fact, there is a simple transformation which maps hierarchical solutions into standard form.

2. The p-hierarchic shape functions possess the following two computational advantages over the nonhierarchic shape functions:

a) possibility of using previous solutions and computations; and,

b) improved conditioning of the equation.

3. Hierarchical shape functions have been developed for triangular elements with C^1 continuity (Ref. 180). However, the form of these functions is fairly complicated for practical applications. In addition, no C^1 shape functions exist for curved-sided elements.

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- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

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